The Laplacian Eigenvalues of Graphs

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Abstract

The Laplacian matrix of a graph is the difference between the diagonal matrix of its vertex degrees and its adjacency matrix. The Laplacian eigenvalues of a graph are those eigenvalues of the associated Laplacian matrix. Laplacian eigenvalues are closely related to almost all major invariants of a graph, linking one extremal property to another. There is no question that Laplacian eigenvalues play a central role in our fundamental understanding of graphs. In the past decades, the Laplacian eigenvalues of graphs have received more and more attention, since they have been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning. In this thesis, we focus on the study of the relationships between Laplacian eigenvalues and structural properties of a graph. Various interesting results on the Laplacian spectral radius, the $k$th largest Laplacian eigenvalue and the algebraic connectivity of a graph are presented. In addition, we investigate some other indices related to the Laplacian eigenvalues of a graph, such as the number of spanning trees, the Laplacian Estrada index, the Laplacian separator and the Laplacian spread. Lastly, we propose some possible directions for further investigating graph Laplacian eigenvalues at the end of this thesis.

Keywords: Graph, adjacency matrix, Laplacian matrix, eigenvalues, spectral radius, Laplacian eigenvalue, Laplacian spectral radius, $k$th largest Laplacian eigenvalue, algebraic connectivity, spanning tree, Laplacian Estrada index, Laplacian separator, Laplacian spread, bound.
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