Abstract

This thesis focuses on optimization problems with generalized orthogonality constraints, which may also contain linear equality constraints. These problems appear in many areas, such as machine learning, signal processing, computer vision and so on.

Many problems in this form are NP hard. One challenge posed by generalized orthogonality constraints is local minimizers loaded by nonconvex constraints. Moreover, the generalized orthogonality constraints are numerically expensive to preserve during iterations.

This thesis is mainly divided into two parts. The first part is focused on solving generalized orthogonality constrained optimization problems with differentiable objective functions. For this class of optimization problems, a generalized gradient flow is proposed, which is contained on the constraints set if the initial condition satisfies generalized orthogonality constraints. The weak convergence of the generalized gradient flow is given. A discrete iterative scheme is also proposed to make the gradient flow method computable. In addition, we analyze the relationship between our discrete iteration scheme and some existing constraint preserving methods, and the relationship between our discrete iteration scheme and the inexact forward-backward method, respectively. Several problems which also can be solved by the generalized gradient flow are given. Furthermore, we also propose an optimal gradient flow by analyzing the first order optimality condition.

The second part of this thesis is devoted to study of the generalized orthogonality constrained optimization problems with nondifferentiable objective functions. An approximate augmented Lagrangian method is used to deal with this class of problems. The global convergence is presented. We also extend the proximal alternating linearized minimization method (EPALM) to deal with the generalized orthogonality constraints appeared in the subproblem of the approximate augmented Lagrangian method. Moreover, to accelerate the EPALM method, an inertial proximal alternat-
ing linearized minimization method (IPALM) is proposed to deal with unconstrained nonconvex, nonsmooth problems with coupled objective functions.

**Keywords:** Generalized Orthogonality Constraints; Stiefel Manifold; Tangent Space; Gradient Flow; Approximate Augmented Lagrangian Method; Proximal Alternating Linearized Minimization Method
Table of Contents

Declaration i

Abstract ii

Acknowledgements iv

Table of Contents v

List of Tables vi

List of Figures vii

Chapter 1 Introduction 1
  1.1 Motivations .................................. 3
  1.2 Contributions ................................ 4
  1.3 Thesis Organization ......................... 5

Chapter 2 Preliminaries 6
  2.1 Notations .................................. 6
  2.2 Stiefel Manifold ............................. 7
    2.2.1 Manifold ................................ 7
    2.2.2 Stiefel Manifold ......................... 9
  2.3 The Gradient Flow Method .................... 13
  2.4 The Augmented Lagrangian Method .......... 15
  2.5 Other Preliminaries ......................... 17
    2.5.1 Preliminaries on Indicator Functions .... 17
    2.5.2 Preliminaries on Kurdyka-Lojasiewicz Property .... 18
    2.5.3 Preliminaries on Inexact Forward-Backward Method .... 19
Chapter 3 The Generalized Gradient Flow Method of Differentiable Optimization Problems

3.1 The Generalized Inner Product and Generalized Orthogonality Constraints

3.1.1 The Generalized Inner Product

3.1.2 Generalized Orthogonality Constraints

3.2 A Generalized Gradient Flow Method of Differentiable Optimization Problems on $\mathcal{P}(p, n)$

3.2.1 Construction of an ODE System

3.2.2 Properties of the Continuous Trajectory

3.2.3 The Discrete Iterative Scheme

3.2.4 Relation to the Inexact Forward-Backward Method

3.3 An Optimal Flow Method of Smooth Optimization Problems on $\mathcal{PK}(p, n)$

3.3.1 Construction of an ODE System

3.3.2 Properties of the Continuous Trajectory

3.3.3 The Discrete Iterative Scheme

3.4 Examples Which can be Solved by the Generalized Gradient Flow

3.4.1 Problems with Linear Equality and Orthogonality Constraints

3.4.2 Problems with Generalized Orthogonality Constraints

3.5 Numerical Experiment on the Generalized Eigenvalue Problem

3.6 Application on the Multiple Graphs Clustering

3.6.1 Background

3.6.2 The Generalized Gradient Flow Method

3.6.3 The Discrete Approximation Scheme

3.6.4 Numerical Results
Chapter 4 The Approximate Augmented Lagrangian Method on Non-smooth Optimization Problems with Generalized Orthogonality Constraints

<table>
<thead>
<tr>
<th>Subsection</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 The Approximate Augmented Lagrangian Method</td>
</tr>
<tr>
<td>4.1.1 The Approximate Augmented Lagrangian Framework</td>
</tr>
<tr>
<td>4.1.2 Convergence Analysis</td>
</tr>
<tr>
<td>4.2 The Extended Proximal Alternating Linearized Minimization Method</td>
</tr>
<tr>
<td>4.2.1 Algorithm and Convergence</td>
</tr>
<tr>
<td>4.2.2 Hybridize EPALM Method with the Approximate Augmented Lagrangian Method</td>
</tr>
<tr>
<td>4.2.3 Numerical Experimentation on Multivariate Data Analysis</td>
</tr>
<tr>
<td>4.3 The Inertial Proximal Alternating Linearized Minimization Method</td>
</tr>
<tr>
<td>4.3.1 Algorithm and Convergence</td>
</tr>
<tr>
<td>4.3.2 An Application on Nonnegative Matrix Factorization Problems</td>
</tr>
</tbody>
</table>

Chapter 5 Summary

<table>
<thead>
<tr>
<th>Subsection</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Summary of Contributions</td>
</tr>
<tr>
<td>5.2 Future Work</td>
</tr>
</tbody>
</table>

References

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>159</td>
</tr>
</tbody>
</table>

Curriculum Vitae

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>171</td>
</tr>
</tbody>
</table>