Some Optimalities of Uniform Designs and
Projection Uniform Designs under Multi-Factor Models

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Preface

In today’s technical and industrial applications, one may face a system with high dimensional input and nonlinear relationship (response) between input and output. Usually, we want to model the complex response by a fitting model based on input values and output ones. Of course, the quality of the fit depends on choices (design) of input values. Due to the complexity of the system, a “space filling” design is needed (e.g., see Fang and Hickernell, 1995, and Bate, Buck, Riccomagno and Wynn, 1996). In the literature some such designs have been proposed. Following are two frequently used designs:

- the uniform design;
- the Latin hypercube sample.

The former supplies global uniform points over the experimental domain in the sense of some model-independent measure of uniformity. While the latter and its versions provide projective uniform sample points in the sense that these points stratify low dimensional margins simultaneously.

The uniform design and the Latin hypercube sample have advantages in computer experiments and their uniformities play important roles in getting these optimalities. Motivated by these facts, this thesis investigates the usefulness of uniformity in experimental designs. The research mainly concentrates on the two kinds of uniformities mentioned above:

- global uniformity;
- projective uniformity.

The measures of usefulness are taken as:

- the criteria in decision theory for approximately linear models and nonparametric models;
- the criteria in optimum design theory for Fourier models and wavelet models.

Since the uniform design is a kind of discrete approximation to the uniform design measure (uniform distribution) over the experimental domain and the most
tables of the uniform designs (see Fang, 1994) are generated by the good lattice
designs, the thesis deals with optimalities of the uniform design measure and the
lattice design. Based on the projective uniformity of the Latin hypercube sample
and its versions, the projective uniform design is introduced and its optimalities in
experimental designs are studied. The whole thesis is organized as follows:

Chapter 1 gives backgrounds and concepts of the uniform design and the Latin
hypercube sample, and introduces optimalities of them in computer experiments,
which are the motivations of the thesis. Finally, a summary of the thesis is given.

Chapter 2 is to make necessary preparations for later use. Some contents related
to optimal design theory, product model and wavelet theory are provided.

Chapter 3 researches optimalities of the uniform design measure for a nonpara-
metric model. Under a framework of decision theory, we prove that the uniform
design measure is an admissible minimax design and the best design among a reason-
able design class. These optimalities and robustness of the uniform design measure
obtained by Wiens (1991) for an approximately linear model also show optimalities
of the uniform design in experimental designs. This is because the uniform design
is a kind of discrete approximation to the uniform design measure.

Chapter 4 studies optimalities of the lattice design for interaction Fourier mod-
els. Riccomagno, Schwabe and Wynn (1995) obtained D-optimal lattice designs for
certain interaction Fourier models. We extend their results to general interaction
Fourier models. These optimalities of the lattice design show at least in part those
of the uniform design as most tables of the uniform designs are generated by the
good lattice designs.

Chapter 5 extends the results of D-optimal designs obtained by Herzberg and
Traves (1994) for Haar wavelet models to interactive Haar type wavelet models. We
prove that a design is D-optimal if and only if it is projective uniform. This result
also shows that the sufficient condition obtained by Herzberg and Traves (1994) is
also necessary.

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Table of Contents

Dedication ................................................................. i
Declaration ................................................................. ii
Preface ................................................................. iii
Acknowledgements ......................................................... v
Table of Contents ......................................................... vi
List of Tables ............................................................. ix
List of Figures ............................................................ x

1. Introduction .............................................................. 1
   1.1 The Challenge of System ............................................. 1
   1.2 Designs for Large Systems .......................................... 7
       1.2a The Uniform Design ............................................ 7
       1.2b A Practical Example of Uniform Design ......................... 9
       1.2c The Latin Hypercube Sample ................................ 12
   1.3 Optimalities of UD and LHS in Computer Experiments .......... 14
       1.3a Optimalities of the Latin Hypercube Sample ................. 15
       1.3b Optimalities of the Uniform Design ......................... 16
   1.4 Summary of the Thesis ............................................ 17
       1.4a Optimality of the Uniform Design Measure ................. 18
       1.4b Optimality of the Lattice-Based Design ..................... 19
       1.4c Optimality of the Projective Uniform Design ............. 19
2. Some Preparations .................................................. 21
  2.1 Optimum Design Theory ......................................... 21
     2.1a The Linear Model ........................................... 21
     2.1b Generalized Designs ......................................... 22
     2.1c Optimal Criteria ............................................ 25
     2.1d Equivalence Theorems ...................................... 30
  2.2 Product Model ................................................... 32
     2.2a Product Interaction .......................................... 32
     2.2b Product Design .............................................. 33
  2.3 Wavelet Theory .................................................. 35
     2.3a Concepts of Wavelets ....................................... 36
     2.3b Multiresolution Analysis ................................... 41
     2.3c Spline-Wavelets ............................................. 46

3. Optimality of the Uniform Design Measure .................. 49
  3.1 Robustness of UDM under an Approximately Linear Model .. 49
  3.2 Criteria of Optimal Designs for a Nonparametric Model .... 52
     3.2a A Frame of the Decision Theory ......................... 53
     3.2b Compare the Frame with that for an Approximately Linear Model .... 57
  3.3 Optimality of UDM under a Nonparametric Model ........... 58
     3.3a Preparatory Work ........................................... 58
     3.3b Admissible Minimaxity of UDM ........................................... 60
     3.3c Optimalities of UDM among a Reasonable Design Subspace ...... 64
  3.4 Some Discussions ............................................... 67

4. Optimality of the Lattice-Based Design ....................... 70
  4.1 Optimality of LBD under a Complete Interaction Fourier Model .... 70
  4.2 Optimality of LBD under a General Interaction Fourier Model .... 75
     4.2a A Class of Interaction Fourier Models ..................... 76
     4.2b Orthogonality ................................................ 80
     4.2c D-Optimality ................................................ 83

5. Optimality of the Projective Uniform Design ............... 85
  5.1 D-Optimal Designs for Haar Wavelet Models .................. 85
  5.2 D-Optimal Designs for $b$-adic Haar Wavelet Models .......... 89
     5.2a D-Optimal Designs for Univariate Models ................. 91